

ON FACTORISATION SYSTEMS FOR SURJECTIVE QUANDLE HOMOMORPHISMS

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ABSTRACT. We study and compare two factorisation systems for surjective homomorphisms in the category of quandles. The first one is induced by the adjunction between quandles and trivial quandles. A precise description of the closure operator on congruences corresponding to this adjunction is provided, which is useful to characterize the two classes of morphisms in this factorisation system. The second one is the factorisation system which has been discovered by E. Bunch, P. Lofgren, A. Rapp and D. N. Yetter.

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INTRODUCTION

A *quandle* is a set A equipped with two binary operations \triangleleft and \triangleleft^{-1} such that the following identities hold:

- $a \triangleleft a = a = a \triangleleft^{-1} a$ for all $a \in A$ (idempotency);
- $(a \triangleleft b) \triangleleft^{-1} b = a = (a \triangleleft^{-1} b) \triangleleft b$ for all $a, b \in A$ (right invertibility);
- $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$ and $(a \triangleleft^{-1} b) \triangleleft^{-1} c = (a \triangleleft^{-1} c) \triangleleft^{-1} (b \triangleleft^{-1} c)$ for all $a, b, c \in A$ (self-distributivity).

This structure was introduced independently in the 1980's by S. V. Matveev [17] and by D. Joyce [16] who first named such a structure a quandle. One of the first goals of this structure was to encode properties of group conjugation in order to find a suitable tool to study objects like the Wirtinger presentation of a knot group. The knot quandle is an invariant allowing one to distinguish two knots up to orientation (see the survey [8] for an introduction to this topic, for instance).

The category \mathbf{Qnd} of quandles and quandle homomorphisms is a *variety* in the sense of universal algebra [5], which is determined by the two operations \triangleleft and \triangleleft^{-1} and the identities recalled here above. It contains the category \mathbf{Qnd}^* of *trivial quandles* as a subvariety, where a quandle is said to be trivial if $\triangleleft = \triangleleft^{-1}$ and $a \triangleleft b = a$ for all $a, b \in A$. Of course, a trivial quandle can be identified with its underlying set, and a quandle homomorphism between trivial quandles is simply a function, so that the category \mathbf{Qnd}^* is isomorphic to the category of sets. This yields an adjunction

$$\begin{array}{ccc} \mathbf{Qnd} & \begin{array}{c} \xrightarrow{\pi_0} \\ \perp \\ \xleftarrow{U} \end{array} & \mathbf{Qnd}^* \end{array} \quad (\text{A})$$

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where $U: \mathbf{Qnd}^* \rightarrow \mathbf{Qnd}$ is the inclusion functor, and $\pi_0: \mathbf{Qnd} \rightarrow \mathbf{Qnd}^*$ is its left adjoint associating with any quandle its set of connected components (see Section 1). The reflector $\pi_0: \mathbf{Qnd} \rightarrow \mathbf{Qnd}^*$ does not preserve all pullbacks, and is not even semi-left-exact in the sense of [9]. Nevertheless, this reflector still has some nice left exactness properties [12], in the sense that it preserves a suitable class of pullbacks inducing an admissible Galois theory in the sense of [15]. This fact guarantees the existence of a canonical factorisation system $(\mathcal{E}, \mathcal{M})$ for surjective homomorphisms of quandles associated with the adjunction (A). We describe this factorisation system in Section 2, by using an important property of permutability of a class of congruences in \mathbf{Qnd} (Lemma 1.4), that is of independent interest. This factorisation system $(\mathcal{E}, \mathcal{M})$ satisfies a characteristic property of the so-called *reflective* ones [9]: \mathcal{E} is the class of surjective homomorphisms which are inverted by the reflector π_0 , and for two composable surjective homomorphisms f and g , then $g \in \mathcal{E}$ whenever $f \circ g \in \mathcal{E}$ and $f \in \mathcal{E}$. The class \mathcal{M} is then the class of trivial extensions (also called trivial coverings) in the sense of categorical Galois theory [15] (see also [10, 11, 12]).

We then describe the so-called *effective closure operator* associated with the adjunction (A). Once again the Lemma 1.4 plays a crucial role, since the permutability of the congruences involved in the construction yields a simple formula for the closure of a congruence.

In the last section we turn our attention to another factorisation system for surjective homomorphisms which was considered by E. Bunch, P. Lofgren, A. Rapp, and D. N. Yetter in [4]. We conclude the article with an example showing that this latter factorisation system, unlike the previous one, does not satisfy the typical property of reflective factorisation systems.

1. THE ADJUNCTION BETWEEN QUANDLES AND TRIVIAL QUANDLES

In this first section, we recall some basic definitions related to the structure of quandles.

Definition 1.1. A *quandle* A is a set equipped with two binary operations $\triangleleft, \triangleleft^{-1}$ satisfying, for all $a, b, c \in A$:

- (A1) $a \triangleleft a = a = a \triangleleft^{-1} a$ (idempotency);
- (A2) $(a \triangleleft b) \triangleleft^{-1} b = a = (a \triangleleft^{-1} b) \triangleleft b$ (right invertibility);
- (A3) $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$ and $(a \triangleleft^{-1} b) \triangleleft^{-1} c = (a \triangleleft^{-1} c) \triangleleft^{-1} (b \triangleleft^{-1} c)$ (self-distributivity).

If A and B are quandles, a function $f: A \rightarrow B$ is a quandle homomorphism when it preserves the operations: $f(a \triangleleft b) = f(a) \triangleleft f(b)$ and $f(a \triangleleft^{-1} b) = f(a) \triangleleft^{-1} f(b)$.

Examples 1.1. (a) Let A be a set, then define $\triangleleft = \triangleleft^{-1}$ as $a \triangleleft b = a$ for all $a, b \in A$. The set A equipped with this operation is a quandle, called *trivial quandle*. We denote by \mathbf{Qnd}^* the category of trivial quandles.

- (b) Let G be a group, define $g \triangleleft h = h^{-1}gh$ and $g \triangleleft^{-1} h = hgh^{-1}$. The set G with these operations is a quandle, called *conjugation quandle*.
- (c) Let G be a group, define $\triangleleft = \triangleleft^{-1}$ with $g \triangleleft h = hg^{-1}h$ for all $g, h \in G$. The set G with this operation is again a quandle, called *core quandle*.
- (d) Let n be a positive integer, define $\triangleleft = \triangleleft^{-1}$ with $i \triangleleft j = 2j - i$ for all $i, j \in \mathbb{Z}_n$. The set \mathbb{Z}_n with this operation defines a quandle, called *dihedral quandle*.

The second and third quandle identities imply that the right actions, denoted by $\rho_b: A \rightarrow A$ and defined by $\rho_b(a) = a \triangleleft b$ for all $a \in A$, are automorphisms (=bijective quandle homomorphisms). We write $\text{Inn}(A)$ for the subgroup of the group $\text{Aut}(A)$ of automorphisms of A generated by all such ρ_b , with $b \in A$. $\text{Inn}(A)$ is called the subgroup of *inner automorphisms* of A . This construction does not lead to a functor, in general, but it behaves functorially with respect to surjective homomorphisms, as shown in [4].

Definition 1.2. A quandle A is *connected* if $\text{Inn}(A)$ acts transitively on A . A *connected component* of A is an orbit under the action of $\text{Inn}(A)$. Two elements a and b of A are in the same orbit if we can find a chain of elements $a_i \in A$ for $1 \leq i \leq n$ that links the elements a and b

$$a \triangleleft^{\alpha_1} a_1 \triangleleft^{\alpha_2} a_2 \cdots \triangleleft^{\alpha_n} a_n = b,$$

where, by convention, one writes

$$a \triangleleft^{\alpha_1} a_1 \cdots \triangleleft^{\alpha_n} a_n := (\dots ((a \triangleleft^{\alpha_1} a_1) \triangleleft^{\alpha_2} a_2) \dots) \triangleleft^{\alpha_n} a_n$$

with $\triangleleft^{\alpha_i} \in \{\triangleleft, \triangleleft^{-1}\}$ for all $1 \leq i \leq n$.

The set of connected components of a quandle A is denoted by $\pi_0(A)$. By looking at this set as a trivial quandle, this determines a functor $\pi_0: \mathbf{Qnd} \rightarrow \mathbf{Qnd}^*$, the left adjoint of the inclusion functor $U: \mathbf{Qnd}^* \rightarrow \mathbf{Qnd}$:

$$\begin{array}{ccc} & \xrightarrow{\pi_0} & \\ \mathbf{Qnd} & \begin{array}{c} \perp \\ \hline \end{array} & \mathbf{Qnd}^* \\ & \xleftarrow{U} & \end{array} \quad (1)$$

We will often drop the full inclusion U from the notations, and write $\eta_A: A \rightarrow \pi_0(A)$ for the A -component of the unit of the adjunction, for example.

The category of quandles is not n -permutable [14] for any n , simply because it contains the category of sets ($\cong \mathbf{Qnd}^*$) as a subvariety. Nevertheless, there is still a large class of congruences which 2-permute, in the sense of composition of relations, with any other congruence on the same quandle. This class of congruences has already been considered in [4], although for a different purpose.

Definition 1.3. If N is a subgroup of $\text{Inn}(A)$, one defines an equivalence relation \sim_N on A by setting: $a \sim_N b$ if and only if a and b lie in the same orbit via the induced action of N on A .

As shown in [4] (Theorem 6.1), the equivalence relation \sim_N is a *congruence* (i.e. \sim_N is also a subalgebra of the algebra $A \times A$) if and only if N is a *normal* subgroup of $\text{Inn}(A)$. The following result concerning the composition of congruences in the category \mathbf{Qnd} is important:

Lemma 1.4. *Let A be a quandle, R a congruence on A , and N a normal subgroup of $\text{Inn}(A)$. Then*

$$\sim_N \circ R = R \circ \sim_N,$$

where

$$\sim_N \circ R = \{(a, b) \in A \times A \mid \exists c \in A \text{ with } (a, c) \in R \text{ and } (c, b) \in \sim_N\}$$

is the relational composite of the congruences \sim_N and R .

Proof. Let $(a, b) \in \sim_N \circ R$, so that there exists $c \in A$ such that aRc and $c \sim_N b$. In particular, there is an element $n \in N$ such that its action c^n on the element c is b , i.e. $c^n = b$. It follows that $(a, b) \in R \circ \sim_N$, since

$$a \sim_N a^n R c^n = b.$$

Accordingly, one has the inclusion $\sim_N \circ R \subset R \circ \sim_N$, and then the equality

$$\sim_N \circ R = R \circ \sim_N .$$

□

Given a surjective homomorphism $f: A \rightarrow B$, we write $\text{Eq}(f)$ for its *kernel congruence* $\text{Eq}(f) = \{(a, b) \in A \times A \mid f(a) = f(b)\}$.

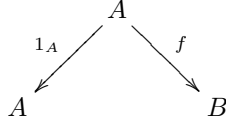
When $\text{Eq}(f) = \sim_N$ with N a normal subgroup of $\text{Inn}(A)$, even though it is not true that $N = \text{Ker}(\text{Inn}(f))$, in general, one always has that $\sim_N = \sim_{\text{Ker}(\text{Inn}(f))}$ (see Theorem 7.1 in [4]). One can then consider the class of those surjective homomorphisms f whose kernel congruence is precisely $\sim_{\text{Ker}(\text{Inn}(f))}$:

$$\mathcal{E}_1 = \{f: A \rightarrow B \mid \text{Eq}(f) = \sim_{\text{Ker}(\text{Inn}(f))}\}.$$

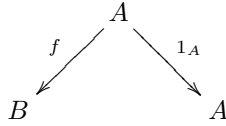
Thanks to Lemma 1.4 we know that the kernel congruences of the arrows in this class have the strong property that they permute with any other congruence on A .

Remark 1.5. Any kernel congruence $\sim_{\text{Inn}(A)}$ of the A -component $\eta_A: A \rightarrow \pi_0(A)$ of the unit η of the adjunction between \mathbf{Qnd} and \mathbf{Qnd}^* belongs to the class \mathcal{E}_1 .

In order to prove a remarkable property of a special type of pushouts in the category \mathbf{Qnd} we need to fix some more notations. Given a homomorphism $f: A \rightarrow B$ one writes f for the relation



representing the graph of f , while the opposite relation



is denoted by f^o . Given a relation R on A and a homomorphism $f: A \rightarrow B$, the direct image of R by f is given by $f(R) = f \circ R \circ f^o$. A homomorphism $f: A \rightarrow B$ is surjective if and only if $f \circ f^o = \Delta_B$, where Δ_B is the equality relation on B . The kernel congruence $\text{Eq}(f)$ of a homomorphism f can be written as the composite $f^o \circ f$.

Lemma 1.4 implies a useful property of a special type of pushouts in \mathbf{Qnd} (see [6] for a general study of permutability of equivalence relations in regular categories).

Lemma 1.6. *Let*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow \bar{g} \\ C & \xrightarrow{\bar{f}} & D \end{array}$$

be a pushout of surjective homomorphisms in \mathbf{Qnd} with the property that $f \in \mathcal{E}_1$. Then the canonical factorisation $(g, f): A \rightarrow C \times_D B$ to the pullback of \bar{f} and \bar{g} is a surjective homomorphism.

Proof. The fact that $f \in \mathcal{E}_1$ implies that

$$\mathbf{Eq}(f) \circ \mathbf{Eq}(g) = \mathbf{Eq}(g) \circ \mathbf{Eq}(f) = \mathbf{Eq}(f) \vee \mathbf{Eq}(g)$$

is the supremum as congruences on A . Moreover, the fact that the square is a pushout implies that $\mathbf{Eq}(t) = \mathbf{Eq}(f) \vee \mathbf{Eq}(g)$, with $t = \bar{f} \circ g$. Consequently,

$$t^o \circ t = f^o \circ f \circ g^o \circ g. \quad (2)$$

The direct image of $(g, f): A \rightarrow C \times_D B$ is given by the composite relation $f \circ g^o$, whereas the relation $(C \times_D B, \pi_1, \pi_2)$ given by the pullback projections is $\bar{g}^o \circ \bar{f}$. Finally, by composing (2) on the left by f and on the right by g^o one obtains the equality

$$f \circ t^o \circ t \circ g^o = f \circ f^o \circ f \circ g^o \circ g \circ g^o$$

so that

$$f \circ f^o \circ \bar{g}^o \circ \bar{f} \circ g \circ g^o = f \circ g^o$$

(since $f \circ f^o = \Delta_B$ and $g \circ g^o = \Delta_C$), and then

$$\bar{g}^o \circ \bar{f} = f \circ g^o,$$

as desired. \square

In particular, the following useful result holds:

Corollary 1.7. *For any surjective homomorphism $f: A \rightarrow B$ in \mathbf{Qnd} the commutative square*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \eta_A \downarrow & & \downarrow \eta_B \\ \pi_0(A) & \xrightarrow{\pi_0(f)} & \pi_0(B) \end{array} \quad (3)$$

where η is the unit of the adjunction (1) has the property that the canonical arrow $(\eta_A, f): A \rightarrow \pi_0(A) \times_{\pi_0(B)} B$ to the pullback (of $\pi_0(f)$ and η_B) is surjective.

Proof. This follows immediately from Lemma 1.6, from Remark 1.5 and the fact that the square (3) is a pushout (this latter property follows from the fact that \mathbf{Qnd}^* is closed under quotients (= surjective homomorphisms) in \mathbf{Qnd}). \square

The following property will also be needed:

Corollary 1.8. *Given a surjective quandle homomorphism $f: A \rightarrow B$, the induced arrow $\bar{f}: \sim_{\text{Inn}(A)} \rightarrow \sim_{\text{Inn}(B)}$ is surjective.*

Proof. By taking the kernel congruences $\sim_{\text{Inn}(A)}$ and $\sim_{\text{Inn}(B)}$ of η_A and η_B in diagram (3), respectively, one gets the following commutative diagram

$$\begin{array}{ccc} \sim_{\text{Inn}(A)} & \xrightarrow{\bar{f}} & \sim_{\text{Inn}(B)} \\ \Downarrow & & \Downarrow \\ A & \xrightarrow{f} & B. \end{array}$$

Let us write $\gamma = (\eta_A, f): A \rightarrow \pi_0(A) \times_{\pi_0(B)} B$ for the induced homomorphism to the pullback, and $p_1: \pi_0(A) \times_{\pi_0(B)} B \rightarrow \pi_0(A)$ and $p_2: \pi_0(A) \times_{\pi_0(B)} B \rightarrow B$ for the pullback projections. To see that the induced homomorphism \bar{f} is surjective, it suffices then to check that the direct image $\gamma(\sim_{\text{Inn}(A)})$ of $\sim_{\text{Inn}(A)}$ along γ is $\text{Eq}(p_1)$, since this will imply that

$$f(\sim_{\text{Inn}(A)}) = (p_2 \circ \gamma)(\sim_{\text{Inn}(A)}) = p_2(\gamma(\sim_{\text{Inn}(A)})) = p_2(\text{Eq}(p_1)) = \sim_{\text{Inn}(B)}.$$

The equality $\gamma(\sim_{\text{Inn}(A)}) = \text{Eq}(p_1)$ follows from the fact that $\gamma \circ \gamma^o = \Delta_{\pi_0(A) \times_{\pi_0(B)} B}$ (since γ is a surjective homomorphism by Corollary (1.7)):

$$\gamma(\sim_{\text{Inn}(A)}) = \gamma \circ \eta_A^o \circ \eta_A \circ \gamma^o = \gamma \circ \gamma^o \circ p_1^o \circ p_1 \circ \gamma \circ \gamma^o = p_1^o \circ p_1 = \text{Eq}(p_1).$$

□

2. CLOSURE OPERATOR ON CONGRUENCES

First let us recall some results about closure operators on congruences, which can be found in [1], in the more general context of regular categories (see also [3]). In the present article we shall consider the special case of varieties of universal algebras, since the category \mathbf{Qnd} is itself a variety.

Definition 2.1. [1] An *idempotent closure operator* $\overline{(\cdot)}$ on congruences in a variety \mathcal{C} consists in giving for every congruence R another congruence \overline{R} called the *closure* of R . This assignment has to satisfy the following properties, where R and S are congruences on A , $f: B \rightarrow A$ is a homomorphism in \mathcal{C} , and $f^{-1}(R)$ is the inverse image of R along f :

- (1) $R \subset \overline{R}$;
- (2) $R \subset S$ implies $\overline{R} \subset \overline{S}$;
- (3) $\overline{f^{-1}(R)} \subset f^{-1}(\overline{R})$;
- (4) $\overline{\overline{R}} = \overline{R}$.

An idempotent closure operator on congruences is called an *effective closure operator* if it also satisfies:

- (5) for any *surjective* homomorphism $f: B \rightarrow A$ one has that $\overline{f^{-1}(R)} = f^{-1}(\overline{R})$.

Let \mathcal{X} be an epireflective subcategory of a variety \mathcal{C} , by which we mean a full reflective subcategory \mathcal{X} of a variety \mathcal{C} , with inclusion $U: \mathcal{X} \rightarrow \mathcal{C}$ and reflector $I: \mathcal{C} \rightarrow \mathcal{X}$, having the property that each component $\eta_A: A \rightarrow I(A)$ of the unit of the adjunction is a surjective homomorphism. Epireflective subcategories of a variety can be characterised in terms of effective closure operators as follows:

Theorem 2.2. [1] *Let \mathcal{C} be a variety. There is a bijection between the epireflective subcategories of \mathcal{C} and the effective closure operators.*

The existence of this bijection was established in Theorem 2.3 in [1]. Given an epireflective subcategory \mathcal{X} of a variety \mathcal{C} with reflector $I: \mathcal{C} \rightarrow \mathcal{X}$, the closure \overline{R} of a congruence R is obtained by taking the canonical quotient $f: A \rightarrow A/R$, and then considering the inverse image $f^{-1}(\text{Eq}(\eta_{A/R}))$ along f of the kernel congruence $\text{Eq}(\eta_{A/R})$ of the A/R -component of the unit η of the adjunction.

In the special case when $\mathcal{X} = \mathbf{Qnd}^*$ is the category of trivial quandles seen as an epireflective subcategory of the variety $\mathcal{C} = \mathbf{Qnd}$ of quandles, the closure \overline{R} of

a congruence R on a quandle A is constructed as the inverse image of the congruence $\text{Eq}(\eta_{A/R}) = \sim_{\text{Inn}(A/R)}$ along the canonical quotient $f: A \rightarrow A/R$:

$$\begin{array}{ccccc}
 & & \overline{R} & \xrightarrow{\quad} & \sim_{\text{Inn}(A/R)} \\
 & \nearrow f_1 & \downarrow & & \downarrow p_1 \\
 R & \xrightarrow{\quad} & A & \xrightarrow{\quad f \quad} & A/R \\
 & \searrow f_2 & & & \downarrow \eta_{A/R} \\
 & & & & \pi_0(A/R)
 \end{array}$$

Equivalently, \overline{R} can be defined as the kernel congruence of the homomorphism $\eta_{A/R} \circ f: A \rightarrow \pi_0(A/R)$. Remark that the kernel congruence $\text{Eq}(\eta_A)$ of the reflection $\eta_A: A \rightarrow \pi_0(A)$ is nothing but the closure $\overline{\Delta}_A$ of the equality relation Δ_A on A . As a consequence, one has that $\overline{\Delta}_A = \sim_{\text{Inn}(A)}$.

The effective closure operator can be described explicitly thanks to Lemma 1.4:

Proposition 2.3. *Let R be a congruence on a quandle A . Then its effective closure \overline{R} in \mathbf{Qnd} corresponding to the reflection (1) is given by*

$$\overline{R} = R \circ \sim_{\text{Inn}(A)}.$$

Proof. Consider the commutative square

$$\begin{array}{ccc}
 A & \xrightarrow{\quad f \quad} & A/R \\
 \eta_A \downarrow & & \downarrow \eta_{A/R} \\
 \pi_0(A) & \xrightarrow{\quad \pi_0(f) \quad} & \pi_0(A/R)
 \end{array}$$

induced by the units of the adjunction (1). As observed above, this square is a pushout. Moreover, the induced factorisation $(\eta_A, f): A \rightarrow \pi_0(A) \times_{\pi_0(A/R)} A/R$ is a surjective homomorphism, as it follows from Corollary (1.7). This implies the following equalities:

$$\overline{R} = R \vee \sim_{\text{Inn}(A)} = R \circ \sim_{\text{Inn}(A)},$$

where $R \vee \sim_{\text{Inn}(A)}$ denotes the supremum of R and $\sim_{\text{Inn}(A)}$ in the lattice of congruences on the quandle A . \square

3. THE INDUCED FACTORISATION SYSTEM FOR SURJECTIVE MORPHISMS

In this section a factorisation system of surjective homomorphisms derived from the reflective subcategory \mathbf{Qnd}^* of \mathbf{Qnd} will be described.

Let us first recall the definition of factorisation system (see [7], for instance, and the references therein).

Definition 3.1. A pair $(\mathcal{E}, \mathcal{M})$ of classes of maps in \mathcal{C} constitutes a *factorisation system* if

- (i) \mathcal{E} and \mathcal{M} contain the identities and are closed under composition with isomorphisms;
- (ii) every map in \mathcal{C} can be written as $m \circ e$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$;

(iii) this factorisation is *functorial*, i.e. given the solid commutative diagram

$$\begin{array}{ccc}
 & \xrightarrow{u} & \\
 e \downarrow & & \downarrow e' \\
 & \xrightarrow{w} & \\
 m \downarrow & & \downarrow m' \\
 & \xrightarrow{v} &
 \end{array}$$

with e, e' in \mathcal{E} and m, m' in \mathcal{M} , there is a unique arrow w making both squares commute.

For instance, any variety of universal algebras \mathcal{C} has the factorisation system defined by the classes \mathcal{E} of surjective homomorphisms and \mathcal{M} of injective homomorphisms.

In this paper we shall be mainly interested in factorisation systems for *surjective homomorphisms* (= quotients) in the category \mathbf{Qnd} of quandles.

The restriction to surjective homomorphisms is related to the fact that the functor π_0 has some particularly nice exactness properties only with respect to the class of surjective homomorphisms in \mathbf{Qnd} : the adjunction (1) is *admissible* in the sense of categorical Galois theory [15], this meaning that the left adjoint $\pi_0 : \mathbf{Qnd} \rightarrow \mathbf{Qnd}^*$ preserves all pullbacks in \mathbf{Qnd} of the form

$$\begin{array}{ccc}
 B \times_{\pi_0(B)} X & \xrightarrow{\pi_2} & X \\
 \pi_1 \downarrow & & \downarrow \phi \\
 B & \xrightarrow{\eta_B} & \pi_0(B)
 \end{array} \tag{3.1}$$

where $\phi: X \rightarrow \pi_0(B)$ is a *surjective* homomorphism in the subcategory \mathbf{Qnd}^* . We include a proof of the following result from [12], based on Corollary (1.7):

Theorem 3.2. *The subvariety \mathbf{Qnd}^* of the variety \mathbf{Qnd} of quandles is admissible.*

Proof. Consider the following commutative diagram where:

- the exterior rectangle is the pullback (3.1), where $\phi: X \rightarrow \pi_0(B)$ is a surjective homomorphism in the subcategory \mathbf{Qnd}^* ;
- the universality of the unit $\eta_{B \times_{\pi_0(B)} X}: B \times_{\pi_0(B)} X \rightarrow \pi_0(B \times_{\pi_0(B)} X)$ induces a unique arrow $\psi: \pi_0(B \times_{\pi_0(B)} X) \rightarrow X$ with $\psi \circ \eta_{B \times_{\pi_0(B)} X} = p_2$;
- the arrow $\gamma: B \times_{\pi_0(B)} X \rightarrow B \times_{\pi_0(B)} \pi_0(B \times_{\pi_0(B)} X)$ is the one induced by the universal property of the pullback of η_B along $\pi_0(p_1)$.

$$\begin{array}{ccccc}
 B \times_{\pi_0(B)} X & \xrightarrow{p_2} & & & X \\
 \downarrow p_1 & \searrow \gamma & \xrightarrow{\eta_{B \times_{\pi_0(B)} X}} & \searrow \psi & \downarrow \phi \\
 & B \times_{\pi_0(B)} \pi_0(B \times_{\pi_0(B)} X) & \xrightarrow{\pi_2} & \pi_0(B \times_{\pi_0(B)} X) & \\
 & \swarrow \pi_1 & & \searrow \pi_0(p_1) & \\
 B & \xrightarrow{\eta_B} & & & \pi_0(B)
 \end{array}$$

By Corollary (1.7), we know that the homomorphism γ is surjective. The fact that $\pi_1 \circ \gamma = p_1$ and $\psi \circ \pi_2 \circ \gamma = p_2$ implies that γ is also injective (since the pullback projections p_1 and p_2 are jointly monomorphic), thus it is an isomorphism.

We can then consider the following diagram

$$\begin{array}{ccccc}
 B \times_{\pi_0(B)} X & \xrightarrow{\eta_B \times_{\pi_0(B)} X} & \pi_0(B \times_{\pi_0(B)} X) & \xrightarrow{\psi} & X \\
 \downarrow p_1 & & \downarrow \pi_0(p_1) & & \downarrow \phi \\
 B & \xrightarrow{\eta_B} & \pi_0(B) & \xrightarrow{1_{\pi_0(B)}} & \pi_0(B)
 \end{array}
 \quad \begin{array}{c} \\ (1) \\ \\ \\ \end{array} \quad \begin{array}{c} \\ (2) \\ \\ \\ \end{array}$$

where both the outer rectangle (1) + (2) and the square (1) are pullbacks. Since η_B is a surjective homomorphism it follows that (2) is a pullback (see Proposition 2.7 in [15], for instance). This shows that the pullback (3.1) is preserved by the functor π_0 , as desired. \square

Remark 3.3. It is not possible to weaken the assumption on $\phi: X \rightarrow \pi_0(B)$, which has to be required to be a *surjective* homomorphism. Indeed, as explained in [12], the functor π_0 does not preserve pullbacks of the form (3.1) when $\phi: X \rightarrow \pi_0(B)$ is not required to be surjective. In other words the functor π_0 is not semi-left-exact [9].

Remark 3.4. One might wonder if, in general, the functor π_0 preserves pullbacks of surjective homomorphisms along surjective homomorphisms. The answer is negative, as the following counter-example shows: π_0 does not even preserve kernel pairs of *split* epimorphisms, in general. This shows a very different behavior of the category **Qnd** with respect to the case of semi-abelian categories, for instance (see [13], Lemma 8.2).

Let us consider the *involution* quandle A (this means that $\triangleleft = \triangleleft^{-1}$) with four elements $\{a, b, c, d\}$ defined by the following table

\triangleleft	a	b	c	d
a	$a \triangleleft a = a$	$a \triangleleft b = a$	$a \triangleleft c = a$	$a \triangleleft d = b$
b	$b \triangleleft a = b$	$b \triangleleft b = b$	$b \triangleleft c = b$	$b \triangleleft d = a$
c	$c \triangleleft a = c$	$c \triangleleft b = c$	$c \triangleleft c = c$	$c \triangleleft d = c$
d	$d \triangleleft a = d$	$d \triangleleft b = d$	$d \triangleleft c = d$	$d \triangleleft d = d$

and the trivial quandle B with two elements $\{x, y\}$. Let $f: A \rightarrow B$ be defined by $f(a) = f(b) = f(c) = x$ and $f(d) = y$. This quandle homomorphism is surjective, and it is even split by the quandle homomorphism $s: B \rightarrow A$ defined by $s(x) = c$ and $s(y) = d$. Its kernel pair $\text{Eq}(f)$ is not preserved by the functor π_0 . Indeed, $[(a, b)]$ and $[(a, a)]$ are distinct elements in $\pi_0(\text{Eq}(f))$ (since (d, d) is the only member of $\text{Eq}(f)$ acting non trivially on $\text{Eq}(f)$), while the corresponding images $([a], [b])$ and $([a], [a])$ are equal in $\text{Eq}(\pi_0(f))$: accordingly, $\text{Eq}(\pi_0(f))$ is not isomorphic to $\pi_0(\text{Eq}(f))$.

Consider the pair $(\mathcal{E}, \mathcal{M})$ of classes of arrows, where \mathcal{E} is given by the surjective homomorphisms inverted by the functor π_0 , and \mathcal{M} is the class of surjective homomorphisms $f: A \rightarrow B$ such that the natural square

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\eta_A \downarrow & & \downarrow \eta_B \\
\pi_0(A) & \xrightarrow{\pi_0(f)} & \pi_0(B)
\end{array} \tag{3.2}$$

induced by the unit η of the adjunction is a pullback. The arrows in the class \mathcal{M} are called *trivial extensions* in categorical Galois theory [15]. Before proving that $(\mathcal{E}, \mathcal{M})$ is a factorisation system for the class of surjective homomorphisms in \mathbf{Qnd} , we first give a description of the morphisms belonging to these two classes in terms of the closure operator on congruences described in the previous section.

Proposition 3.5. *A surjective homomorphism $f: A \rightarrow B$ belongs to the class \mathcal{E} if and only if $\mathbf{Eq}(f) \subset \sim_{\text{Inn}(A)}$ if and only if $\overline{\mathbf{Eq}(f)} = \mathbf{Eq}(f) \circ \sim_{\text{Inn}(A)} = \sim_{\text{Inn}(A)}$.*

Proof. The fact that π_0 inverts a surjective homomorphism $f: A \rightarrow B$ obviously implies that $\mathbf{Eq}(f) \subset \sim_{\text{Inn}(A)}$.

Conversely, suppose now that $\mathbf{Eq}(f) \subset \sim_{\text{Inn}(A)}$, so that we have the following commutative diagram

$$\begin{array}{ccccc}
& & \sim_{\text{Inn}(A)} & \xrightarrow{\bar{f}} & \sim_{\text{Inn}(B)} \\
& \nearrow & \downarrow & & \downarrow \\
& & A & \xrightarrow{f} & B \\
& \nearrow & \downarrow \eta_A & & \downarrow \eta_B \\
\mathbf{Eq}(f) & \xRightarrow{\quad} & \pi_0(A) & \xrightarrow{\pi_0(f)} & \pi_0(B)
\end{array}$$

(Note: In the original image, there are additional arrows: a dotted arrow from $\mathbf{Eq}(f)$ to $\sim_{\text{Inn}(A)}$, a dotted arrow from $\pi_0(A)$ to B labeled ϕ , and two vertical arrows from $\sim_{\text{Inn}(A)}$ and $\sim_{\text{Inn}(B)}$ to A and B respectively, labeled p_1 and p_2 .)

where the induced dotted homomorphism \bar{f} is a surjective homomorphism ($f(\sim_{\text{Inn}(A)}) = \sim_{\text{Inn}(B)}$ by Corollary (1.8)), and the induced dotted homomorphism ϕ is such that $\phi \circ f = \eta_A$. It follows that $\phi \circ p_1 = \phi \circ p_2$, and there exists then a unique morphism $\psi: \pi_0(B) \rightarrow \pi_0(A)$ with $\psi \circ \eta_B = \phi$, which is the inverse of $\pi_0(f)$.

Finally, the equivalence with the condition $\overline{\mathbf{Eq}(f)} = \mathbf{Eq}(f) \circ \sim_{\text{Inn}(A)} = \sim_{\text{Inn}(A)}$ immediately follows from Proposition 2.3. \square

Proposition 3.6. *A surjective homomorphism $f: A \rightarrow B$ belongs to the class \mathcal{M} if and only if $\mathbf{Eq}(f) \cap \sim_{\text{Inn}(A)} = \Delta_A$.*

Proof. This follows immediately from Corollary (1.7). Indeed, given the commutative square (3.2), we know that the factorisation $(\eta_A, f): A \rightarrow \pi_0(A) \times_{\pi_0(B)} B$ is a surjective homomorphism, which will be also injective if and only if its kernel congruence $\mathbf{Eq}(f) \cap \overline{\Delta_A}$ is the identity relation Δ_A on A . \square

We now show that any surjective homomorphism $f: A \rightarrow B$ has an $(\mathcal{E}, \mathcal{M})$ factorisation:

Proposition 3.7. *Let $f: A \rightarrow B$ be a surjective homomorphism in \mathbf{Qnd} , then it has a factorisation $\tilde{f} \circ p$, where p belongs to \mathcal{E} and \tilde{f} belongs to \mathcal{M} .*

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc}
 \text{Eq}(f) \cap \sim_{\text{Inn}(A)} & \longrightarrow & \sim_{\text{Inn}(A)} & & \\
 \downarrow & & \Downarrow & & \\
 \text{Eq}(f) & \xrightarrow{\quad\quad\quad} & A & \xrightarrow{f} & B \\
 \vdots & & \searrow p & & \nearrow \tilde{f} \\
 \text{Eq}(\tilde{f}) & \xrightarrow{\quad\quad\quad} & \text{Eq}(f) \cap \sim_{\text{Inn}(A)} & &
 \end{array}$$

where p is the canonical quotient, and \tilde{f} is the unique homomorphism such that $\tilde{f} \circ p = f$. Note that by construction $\text{Eq}(p) \subset \sim_{\text{Inn}(A)}$, so that p is a surjective homomorphism inverted by π_0 . Furthermore, one has the equalities

$$\Delta_{\frac{A}{\text{Eq}(f) \cap \sim_{\text{Inn}(A)}}} = p(\text{Eq}(f) \cap \sim_{\text{Inn}(A)}) = \text{Eq}(\tilde{f}) \cap \sim_{\text{Inn}(\frac{A}{\text{Eq}(f) \cap \sim_{\text{Inn}(A)}})},$$

where $p(\sim_{\text{Inn}(A)}) = \sim_{\text{Inn}(\frac{A}{\text{Eq}(f) \cap \sim_{\text{Inn}(A)}})}$ thanks to Corollary 1.8. By Proposition 3.6 it follows that \tilde{f} belongs to \mathcal{M} . \square

We are now ready to show that the classes of surjective homomorphisms

$$\mathcal{E} = \{f: A \rightarrow B \mid \text{Eq}(f) \subset \overline{\Delta}_A\}$$

and

$$\mathcal{M} = \{f: A \rightarrow B \mid \text{Eq}(f) \cap \overline{\Delta}_A = \Delta_A\}$$

form a factorisation system in the category $\mathbf{Qnd}_{\text{RegEpi}}$ of quandles and surjective homomorphisms (= regular epimorphisms in \mathbf{Qnd}):

Proposition 3.8. $(\mathcal{E}, \mathcal{M})$ is a factorisation system in $\mathbf{Qnd}_{\text{RegEpi}}$.

Proof. The condition (i) in Definition 3.1 is easily checked, while condition (ii) is guaranteed by Proposition 3.7. To check the condition (iii) in the definition of a factorisation system it suffices to show that the two classes \mathcal{E} and \mathcal{M} are orthogonal: for any commutative diagram

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 g \downarrow & & \downarrow h \\
 C & \xrightarrow{m} & D
 \end{array} \tag{3.3}$$

in \mathbf{Qnd} where $f \in \mathcal{E}$ and $m \in \mathcal{M}$, we have to prove the existence of a unique morphism $t: B \rightarrow C$ such that $t \circ f = g$ and $m \circ t = h$. Consider the cube

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & B & & \\
 \eta_A \searrow & & \eta_B \searrow & & \\
 & \pi_0(A) & \xleftarrow{\varphi} & \pi_0(B) & \\
 & \downarrow \varphi^{-1} & & \downarrow \pi_0(h) & \\
 C & \xrightarrow{m} & D & & \\
 \eta_C \searrow & & \eta_D \searrow & & \\
 & \pi_0(C) & \xrightarrow{\pi_0(m)} & \pi_0(D) & \\
 & \downarrow \pi_0(g) & & \downarrow & \\
 & & & &
 \end{array}$$

where the bottom face is a pullback since m belongs to \mathcal{M} , and $\pi_0(f) = \varphi$ is an isomorphism since $f \in \mathcal{E}$. The universal property of the pullback and the equality

$$\pi_0(m) \circ \pi_0(g) \circ \varphi^{-1} \circ \eta_B = \pi_0(h) \circ \eta_B = \eta_D \circ h$$

induce a unique $t: B \rightarrow C$ such that, in particular, $m \circ t = h$. The equality $t \circ f = g$ follows from the fact that the morphisms η_C and m are jointly monomorphic. \square

4. COMPARISON WITH ANOTHER FACTORISATION SYSTEM

Finally, let us compare this factorisation system with another one in $\mathbf{Qnd}_{\text{RegEpi}}$. In [4], E. Bunch, P. Lofgren, A. Rapp and D. N. Yetter showed that every surjective homomorphism in \mathbf{Qnd} has a canonical factorisation whose second component is what the authors of that article call a *rigid quotient*, namely a surjective homomorphism h such that $\text{Inn}(h)$ is an isomorphism.

Proposition 4.1. *Let $f: A \rightarrow B$ be a surjective homomorphism in \mathbf{Qnd} . Then f has a factorisation as $f = h \circ g$, where $g: A \rightarrow A / \sim_{\text{Ker}(\text{Inn}(f))}$ and $h: A / \sim_{\text{Ker}(\text{Inn}(f))} \rightarrow B$ is such that $\text{Inn}(h)$ is an isomorphism.*

Thanks to the result in [4], we now show that the classes of surjective homomorphism

$$\mathcal{E}_1 = \{f: A \rightarrow B \mid \text{Eq}(f) = \sim_{\text{Ker}(\text{Inn}(f))}\}$$

and

$$\mathcal{M}_1 = \{f: A \rightarrow B \mid \text{Inn}(f) \text{ is an isomorphism}\}$$

form a factorisation system for surjective homomorphisms:

Proposition 4.2. *$(\mathcal{E}_1, \mathcal{M}_1)$ is a factorisation system in $\mathbf{Qnd}_{\text{RegEpi}}$.*

Proof. The first axiom in the definition of factorisation system is easy to check, while (ii) is precisely Theorem 8.1 in [4] (recalled as Proposition 4.1 here above). To check the validity of property (iii) consider a commutative square of surjective homomorphisms (3.3) with $f \in \mathcal{E}_1$ and $m \in \mathcal{M}_1$. By applying the functor Inn to this commutative square we get the commutative diagram of surjective homomorphisms

of groups

$$\begin{array}{ccccc}
 & & \text{Ker}(\text{Inn}(g)) & & \\
 & \nearrow \iota & \downarrow k & & \\
 \text{Ker}(\text{Inn}(f)) & \xrightarrow{k'} & \text{Inn}(A) & \xrightarrow{\text{Inn}(f)} & \text{Inn}(B) \\
 & & \downarrow \text{Inn}(g) & & \downarrow \text{Inn}(h) \\
 & & \text{Inn}(C) & \xrightarrow{\text{Inn}(m)} & \text{Inn}(D)
 \end{array}$$

with $\text{Inn}(m)$ an isomorphism. Accordingly, there is an induced inclusion $\iota: \text{Ker}(\text{Inn}(f)) \hookrightarrow \text{Ker}(\text{Inn}(g))$ between the kernels such that $k \circ \iota = k'$. This induces an inclusion $\iota': \sim_{\text{Ker}(\text{Inn}(f))} \rightarrow \sim_{\text{Ker}(\text{Inn}(g))}$ of the corresponding kernel congruences in \mathbf{Qnd} . Using Proposition 4.1, one obtains an $(\mathcal{E}_1, \mathcal{M}_1)$ factorisation $\tilde{h} \circ \tilde{g}$ of g as in the diagram

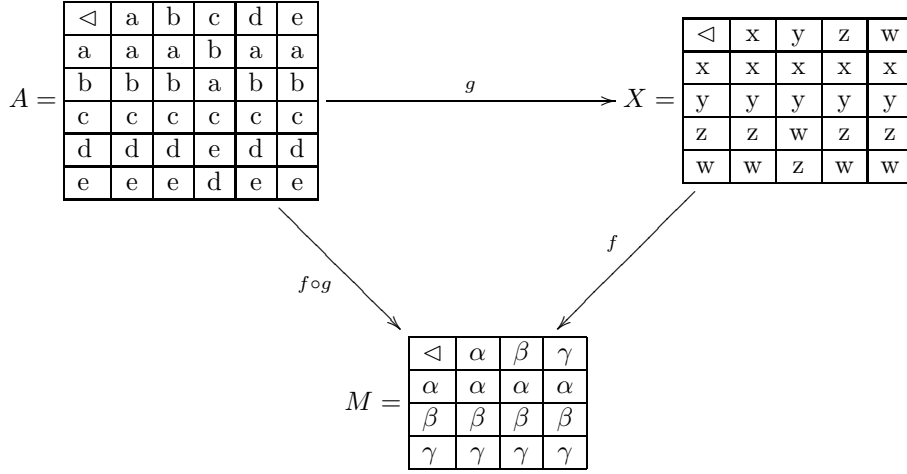
$$\begin{array}{ccccc}
 & & \sim_{\text{Ker}(\text{Inn}(g))} & & \\
 & \nearrow & \downarrow & & \\
 \sim_{\text{Ker}(\text{Inn}(f))} & \xRightarrow{\quad} & A & \xrightarrow{f} & B \\
 & & \downarrow g & \searrow \tilde{g} & \downarrow h \\
 & & C & \xrightarrow{\tilde{h}} & D \\
 & & & \nearrow \phi & \\
 & & & A / \sim_{\text{Ker}(\text{Inn}(g))} &
 \end{array}$$

This induces a homomorphism $\phi: B \rightarrow A / \sim_{\text{Ker}(\text{Inn}(g))}$ such that $\phi \circ f = \tilde{g}$. The arrow $\tilde{h} \circ \phi$ is the desired factorisation showing the orthogonality of \mathcal{E}_1 and \mathcal{M}_1 . \square

By comparing this factorisation system with the one considered in the previous section one remarks that $\mathcal{E}_1 \subset \mathcal{E}$, since $\text{Ker}(\text{Inn}(f)) \subset \text{Inn}(A)$ and, consequently, $\mathcal{M} \subset \mathcal{M}_1$.

Remark 4.3. We finally observe that the factorisation system $(\mathcal{E}_1, \mathcal{M}_1)$ does not have the property that g belongs to \mathcal{E}_1 whenever $f \circ g$ and f belong to \mathcal{E}_1 . This shows a difference with the factorisation system $(\mathcal{E}, \mathcal{M})$ in $\mathbf{Qnd}_{\text{RegEpi}}$ considered in the previous section (the class \mathcal{E} obviously satisfies this property).

Consider the following commutative diagram of involutive quandles:



Let $g: A \rightarrow X$ defined by $g(a) = g(b) = x$, $g(c) = y$, $g(d) = z$ and $g(e) = w$ and let $f: X \rightarrow M$ defined by $f(x) = \alpha$, $f(y) = \beta$ and $f(z) = f(w) = \gamma$. One can show that $f = \eta_X$ and $f \circ g = \eta_A$ so that both f and $g \circ f$ are in \mathcal{E}_1 . To see that g is not in \mathcal{E}_1 , remark that $(a, b) \in \text{Eq}(g)$ but the only member of $\text{Inn}(A)$ linking them is ρ_c which does not belong to $\text{Ker}(\text{Inn}(g))$ (since $\text{Inn}(g)(\rho_c) = \rho_y$).

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